MATH 2050 - Continuity of functions

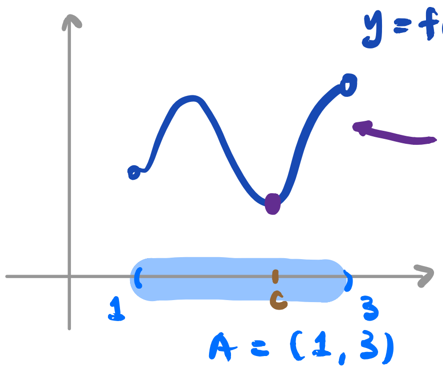
(Reference: Bartle §5.1, 5.2)

Q: What does "continuity" mean?

$f : A \to \mathbb{R}$

A: "f is continuous at c"

$\iff$ "$f(x) \underset{\varepsilon}{\approx} f(c)$ when $x \underset{\delta}{\approx} c$"



This is "continuous".

Note: We NEED $c \in A$.

Def^{n}: (ε-δ def^{n} for continuity)

Given $f : A \to \mathbb{R}$ and $c \in A$, we say that "f is continuous (cts) at c" if $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t.

(*) ...... $|f(x) - f(c)| < \varepsilon$ whenever $x \in A$, $|x - c| < \delta$

Remark: Compared to the def^{n} of $\lim\limits_{x \to c} f(x) = L$, we have

- L is replaced by $f(c)$ $\Rightarrow$ $c \in A$
- $f(c)$ matters here, unlike $\lim\limits_{x \to c} f(x) = L$
- (*) is always satisfied at $x = c$
- c may or may not be a cluster point of A

Note: Continuity of f at $c \in A$ is sensitive to the value of $f(c)$.

For the last remark,

Case 1: When c IS a cluster pt. of A

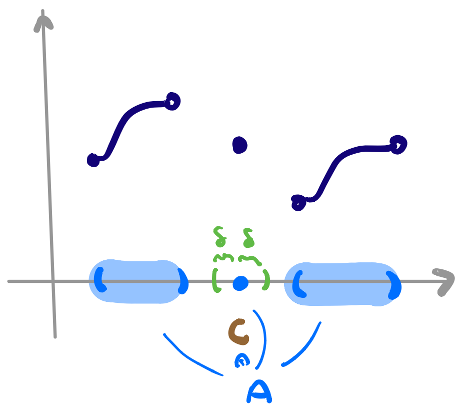
"f is cts at $c \in A$" $\Leftrightarrow$ "$\lim_{x \to c} f(x) = f(c)$"

interesting case

i.e. You can "substitute" to evaluate the limit at c.

Case 2: When c is NOT a cluster pt. of A

Then, f is always cts at $c \in A$



Why? In this case, $\exists\, \delta > 0$ s.t.

$$A \cap (c-\delta, c+\delta) = \{c\}$$

$\Rightarrow$ $(*)$ is trivially satisfied.

Note: "continuity" is a pointwise condition.

Def^n: $f: A \to \mathbb{R}$ is continuous on a subset $B \subseteq A$ if f is continuous at EVERY $c \in B$.

In particular, if $B = A$, then we say f is continuous (everywhere).

Examples of continuous functions

- $f(x) = b$ constant function
- $f(x) = x$ or $f(x) = x^2$
- $f(x) = p(x)$ polynomial function
- $f(x) = \sin x$ or $\cos x$ or $\tan x$
- $f(x) = e^x$ or $\sqrt{x}$

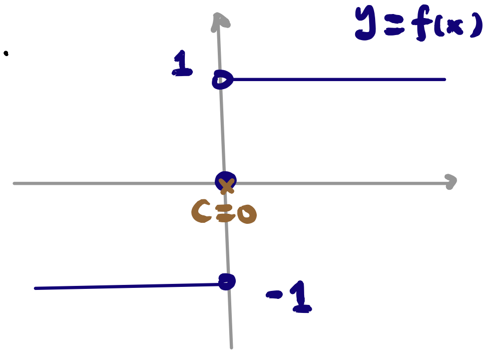
Example of dis-continuous functions

Example 1: Consider $f: \mathbb{R} = A \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

"sign function"

Show that f is NOT cts at $x = 0$.



Proof: Note $0 \in A$ is a cluster pt. of $A = \mathbb{R}$.

Check whether $\lim_{x \to 0} f(x) \stackrel{?}{=} f(c)$

In this case $\lim_{x \to 0} f(x)$ DOES NOT EXIST!

Consider $(x_n) = \left(\frac{(-1)^n}{n}\right) \to 0$ and

note $(f(x_n)) = ((-1)^n)$ is divergent

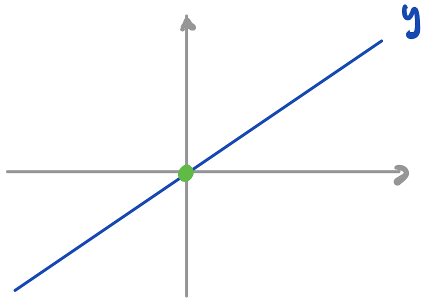
$\xRightarrow[\text{Criteria}]{\text{Seq}}$ $\lim_{x \to 0} f(x)$ does not exist.

$\blacksquare$

Remark: For this f, it is discontinuous at 0 no matter what the value of $f(0)$ is.

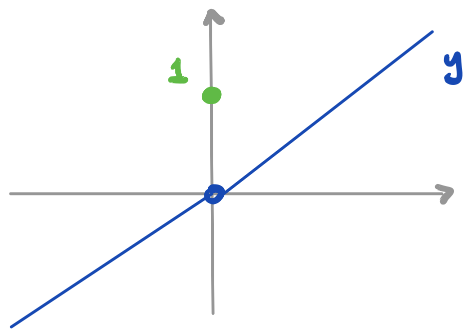
Example: [Sometimes you can make a fcn cts by redefining it at a pt.]

$$f(x) := \begin{cases} x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \xleftrightarrow[\text{only at } x=0]{\text{differs}} \quad g(x) := \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$



BUT:

$\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} g(x)$



This is cts at 0.

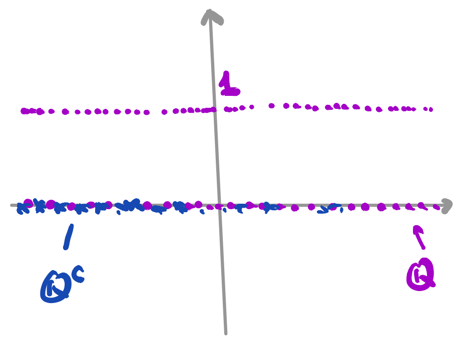
This is NOT cts at 0.

Example 2: The function $f: A = \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is dis-continuous EVERYWHERE.

Proof: (#) Key idea: Density of $\mathbb{Q}$ or $\mathbb{Q}^c$ in $\mathbb{R}$.



Take $c \in \mathbb{R}$. There are 2 cases:

Case 1: $c \in \mathbb{Q}$.

Claim: $\lim_{x \to c} f(x)$ DOES NOT EXIST.

Reason: $\exists$ rational numbers $(x_n) \to c$ with $x_n \neq c$ (#) $\Rightarrow (f(x_n)) = (1) \to 1$

$\exists$ irrational numbers $(x_n') \to c$ with $x_n' \neq c$ (#) $\Rightarrow (f(x_n')) = (0) \to 0$

density (#)

DONE by Seq. criteria!

Case 2: $c \notin \mathbb{Q}$ is the same. $\blacksquare$

Q: How to construct NEW cts fcn from OLD ones?

A: "most of the time" use limit theorems.

Thm 1: $f, g: A \to \mathbb{R}$ is cts (at $c \in A$)

$\Rightarrow f \pm g,\ fg,\ f/g$ is cts (at $c \in A$) wherever they are defined

$g(x) = x$ cts everywhere $\rightsquigarrow$ $\frac{1}{g}(x) = \frac{1}{x}$ cts everywhere it is defined, i.e. $x \neq 0$

Thm 2: $f: A \to \mathbb{R}$ is cts (at $c \in A$)

$\Rightarrow$ $\sqrt{f}$, $|f|$ are cts (at $c \in A$) wherever they are defined.

Thm 3: (Composition of functions)

If f is cts at $c \in A$, and

g is cts at $f(c) \in B$,

then $g \circ f$ is cts at $c \in A$.

$f: A \to \mathbb{R}$

$g: B \to \mathbb{R}$

and $f(A) \subseteq B$

$\Rightarrow g \circ f : A \to \mathbb{R}$

$g \circ f(x) := g(f(x))$

Proof: "Use ε-δ def^{n}". Let $b := f(c) \in B$

Let $\varepsilon > 0$ be fixed but arbitrary.

Since g is cts at $b = f(c)$, then $\exists\ \delta_1 = \delta_1(\varepsilon) > 0$ s.t.

(†) ...... $|g(y) - g(b)| < \varepsilon$ when $y \in B$, $|y - b| < \delta_1$

new ε

Since f is cts at $c \in A$, for the $\delta_1 > 0$, $\exists\ \delta_2 = \delta_2(\delta_1) > 0$ s.t.

(††) ...... $|f(x) - f(c)| < \delta_1$ when $x \in A$, $|x - c| < \delta_2$

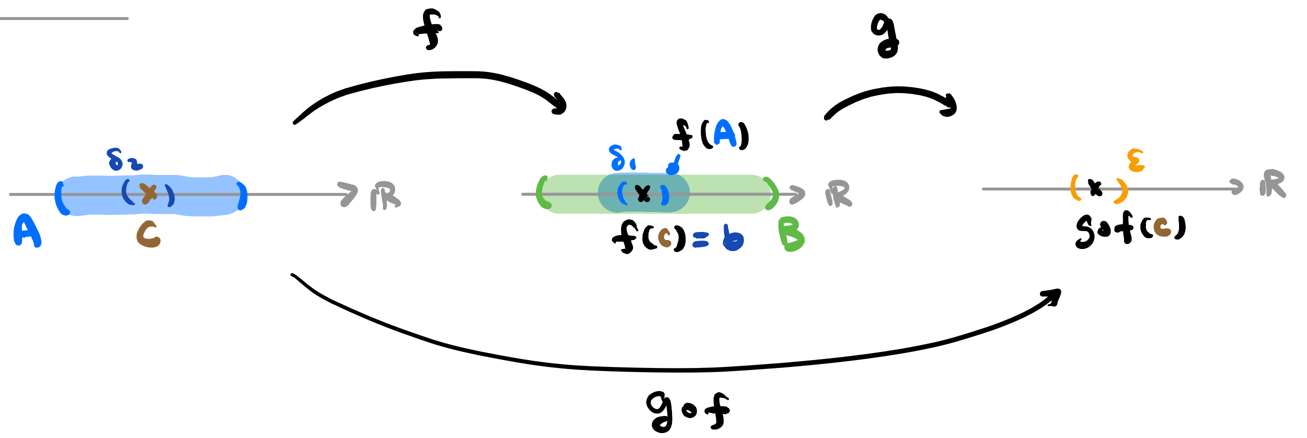
For such $\delta_2 > 0$, when $x \in A$, $|x - c| < \delta_2$

by (++), $\quad |\underbrace{f(x)}_{y} - \underbrace{f(c)}_{b:=}| < \delta_1$

by (+), $\quad |\underbrace{g(f(x))}_{g\circ f(x)} - \underbrace{g(f(c))}_{g\circ f(c)}| < \varepsilon$

$\square$

Picture:



Exercise: Prove this using sequential criteria.